Imperial College London

Lecture 9

Combining Signals & Systems together using IMU as an example

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7 things you have learned about systems (1)

1. A system can be characterised in terms of **differential equations** in the time-domain, or in terms of **transfer functions** in the Laplace domain.

2. **Laplace transform** converts differential equations into **algebraic equations**. While in Fourier, we use the frequency variable $j\omega$, in Laplace, we use the complex frequency variable s, where $s = \alpha + j\omega$.

3. **Transfer function** H(s) is the relationship between output Y(s) and input X(s) in the Laplace domain:

$$Y(s) = H(s) \times X(s)$$

4. **Frequency response** $H(j\omega)$ = Transfer function H(s) evaluated at $s = j\omega$:

$$H(j\omega) = H(s)\Big|_{s=j\omega}$$

7 things you have learned about systems (2)

5. Fourier transform and frequency response are only valid of **steady-state conditions**; Laplace transform and transfer function are useful for **both steady state and transient conditions**.

6. **Non-linear systems** can be approximated as linear if you operate over a small signal range.

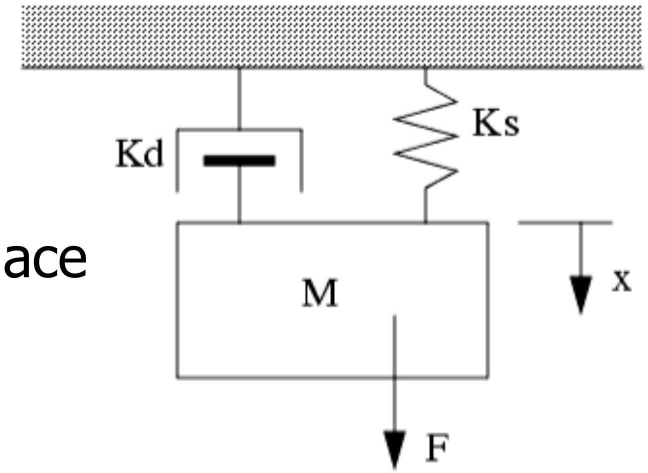
7. A general **2^{nd} order system** can be expressed in term of **damping factor** ζ and **resonant frequency** ω_0, and it can be under-damped, critically- damped or over-damped.

Topics omitted:

- Ideas of poles and zeroes of a system
- How poles and zeroes affect steady-state and transient behaviour of a system
- Stability issues of a system

1 & 2 - Differential Equation vs Transfer Function

- We use differential equations to model systems in time-domain:



$$M\ddot{x}(t) + K_d\dot{x}(t) + K_s x(t) = F(t)$$

- We use transfer functions to model systems in Laplace domain as algebraic equation:

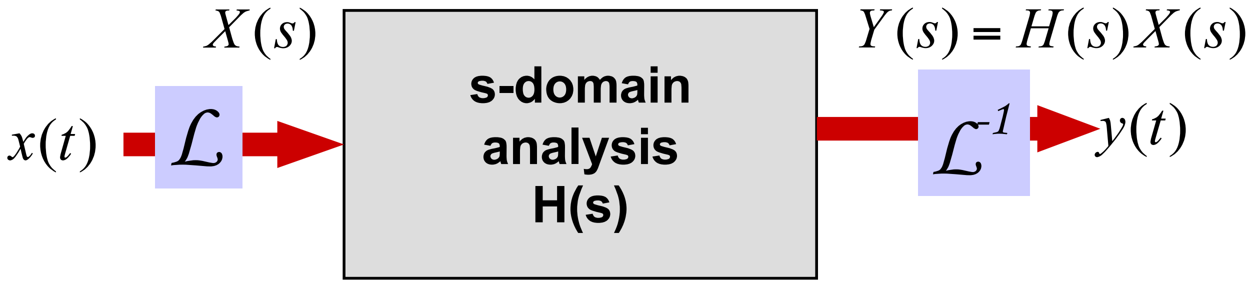
$$Ms^2X(s) + K_d sX(s) + K_s X(s) = F(s)$$

$$(Ms^2 + K_d s + K_s)X(s) = F(s)$$

$$\Longrightarrow H(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + K_d s + K_s}$$

$$\boxed{s^k \longleftrightarrow \frac{d^k}{dt^k}}$$

3 - Transfer Function H(s) = Y(s) / X(s)



- Once transformed to the s-domain, analysis and prediction of the system become easy if we know the system's characteristic H(s), which is also called the **transfer function**.

- Transfer function H(s) = Output Y(s) / Input X(s)

or

Output Y(s) = Transfer Function H(s) x Input X(s)

4 – Frequency Response vs Transfer Function

Laplace Transform

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)e^{-st}dt$$

$s = j\omega$ ⇒

Fourier Transform

$$\mathcal{F}\{x(t)\} = X(j\omega) = \int_0^\infty x(t)e^{-j\omega t}dt$$

Transfer Function

$$H(s)$$

$s = j\omega$ ⇒

Frequency Response

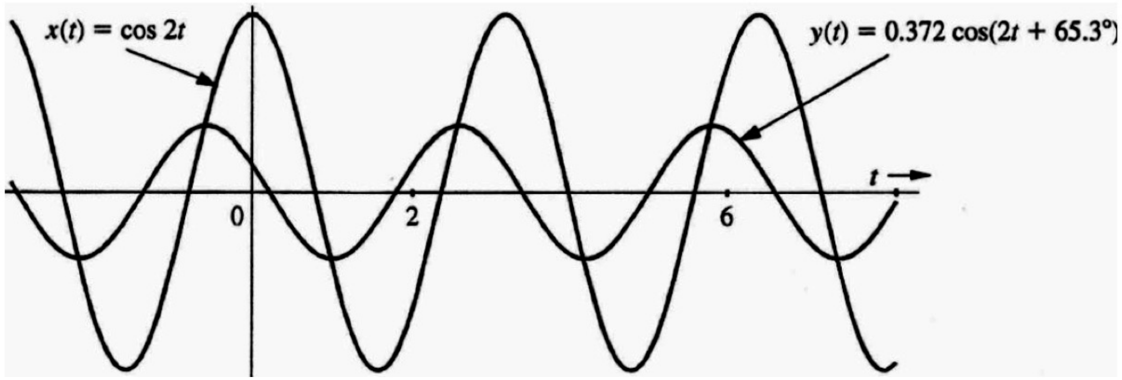
$$H(j\omega)$$

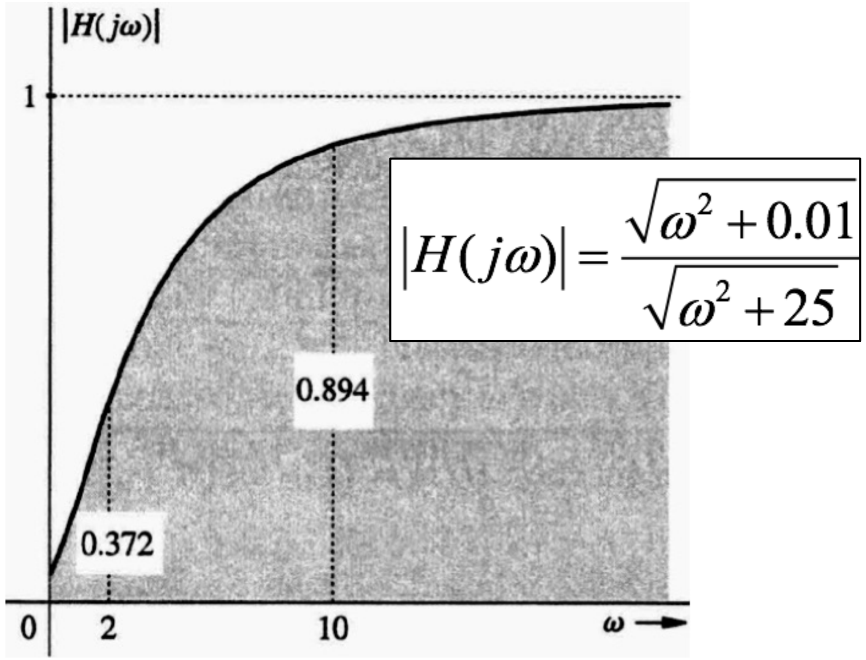
- We can find the frequency response of a system by substituting s = jω into the transfer function:

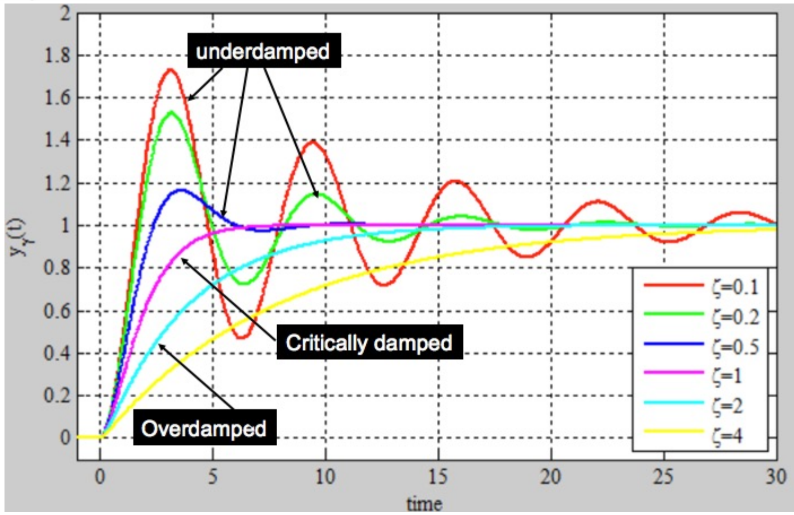
$$H(s)\Big|_{s=j\omega} = H(j\omega) = |H(j\omega)|\, e^{j\,\angle H(j\omega)}$$

5 – Steady state vs Transient

- When using frequency response H(jω), we assume inputs are everlasting sinusoids at frequency jω. That is, we assume that all transient conditions have died down.



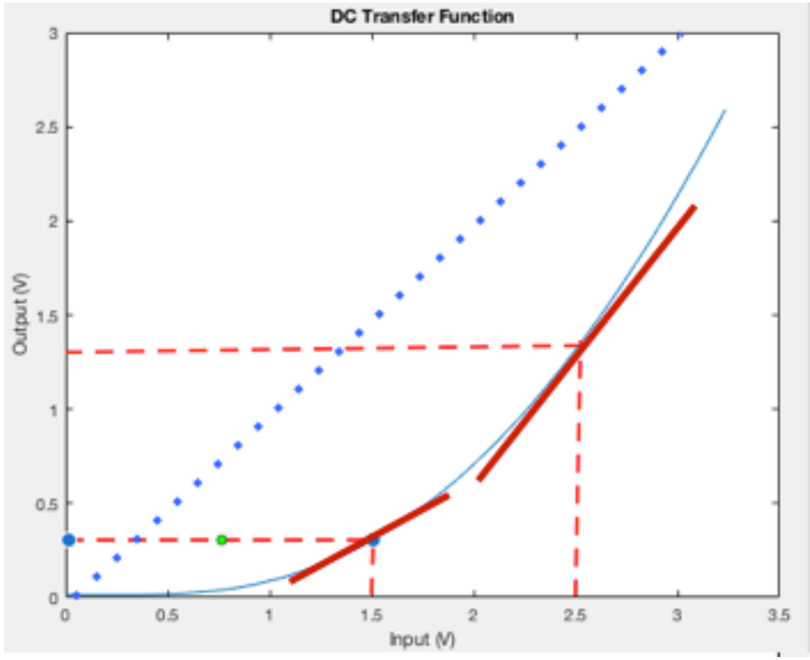


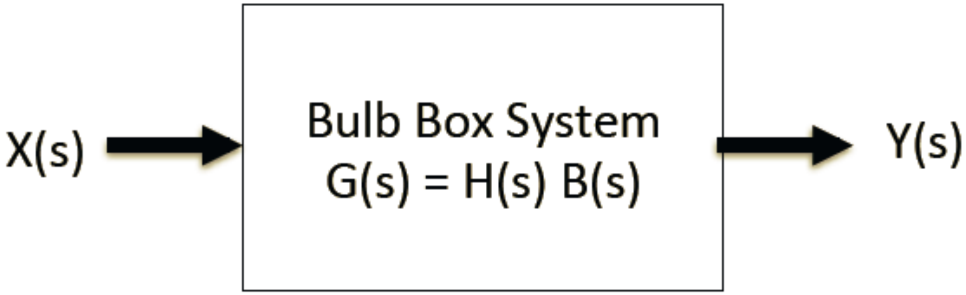


- When using Laplace transform to model system behaviour, we can model both steady state and transient behaviours.

6 – Linear approximation in small signals

- A non-linear system such as the Bulb Board can be approximated as linear provided that we only **operate** in small region.
- We called this the "**operating point**" of the system.
- Shown here are two operating points, one at 2.5V input, and another at 1.5V input.
- If we now only use small signal amplitude (sinewaves), then the behaviour of the non-linear system is approximated to be linear.
- Now, we can use transfer functions to model its behaviour as shown here.





$$G(s) = \frac{1000}{0.038s^3 + 1.19s^2 + 43s + 1000}$$

7 – Damping factor and Natural frequency

- Let us take the transfer function H(s) of the 2nd order system used in Bulb Board is an example:

  - $\omega_0 = \sqrt{a_0} = 31.62$, **resonant frequency** in rad/sec, or 31.62/2π = 5Hz
  - $\zeta = \frac{a_1}{2\sqrt{a_0}} = \frac{5}{2\sqrt{1000}} = 0.079$, the **damping factor** (very small, ideal = 1)
  - $K = \frac{b_0}{a_0} = 1$, DC gain of the system at zero frequency

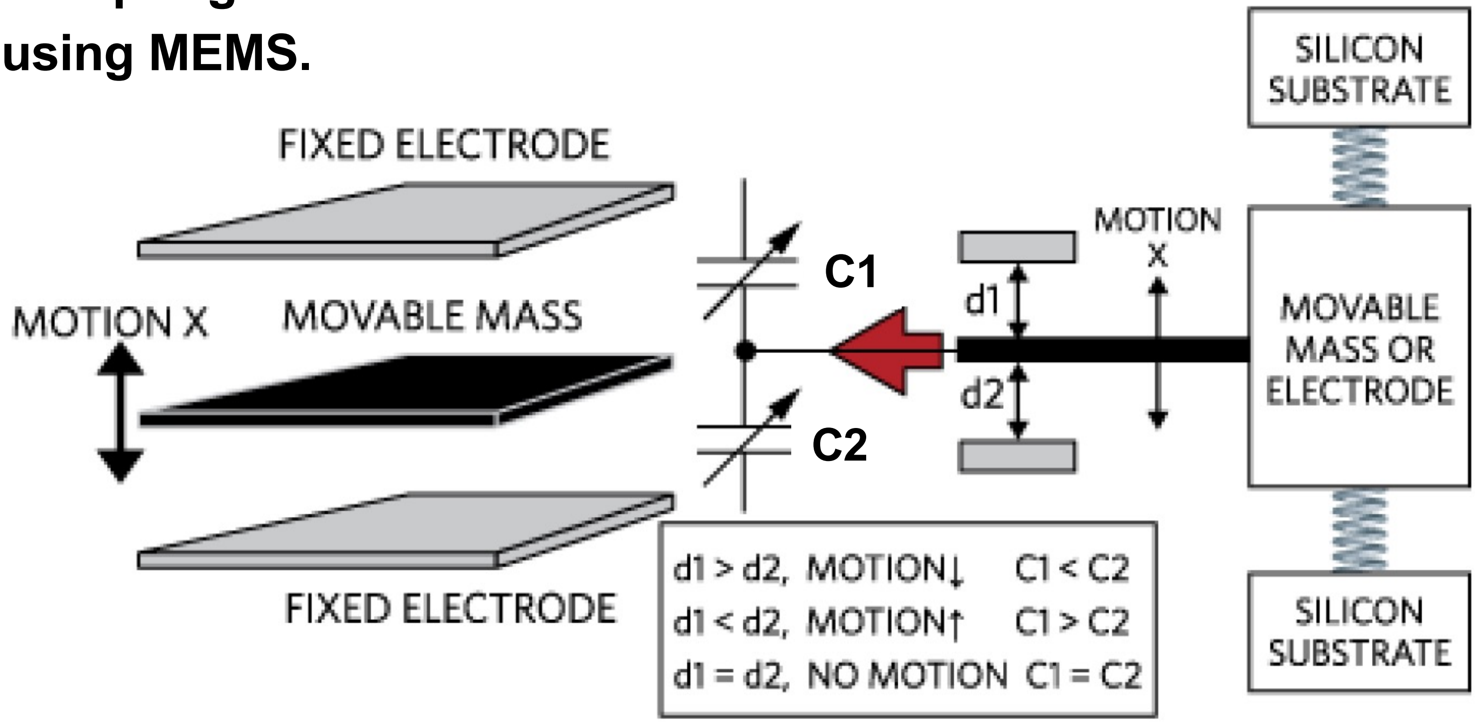
- Since the damping factor is very small (much smaller than 1), this system is highly oscillatory.

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_0} = K\frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2}$$

Motion Sensing – Accelerometer

Basic Principle

- Newton's 2nd Law of motion: F = mass x acceleration.
- Sense acceleration is really sensing the force on a mass.
- Use capacitive sensing with MEMS.
- Acceleration causes mass to move.
- Mass pivoted on springs anchored one side as shown.
- Implemented using MEMS.

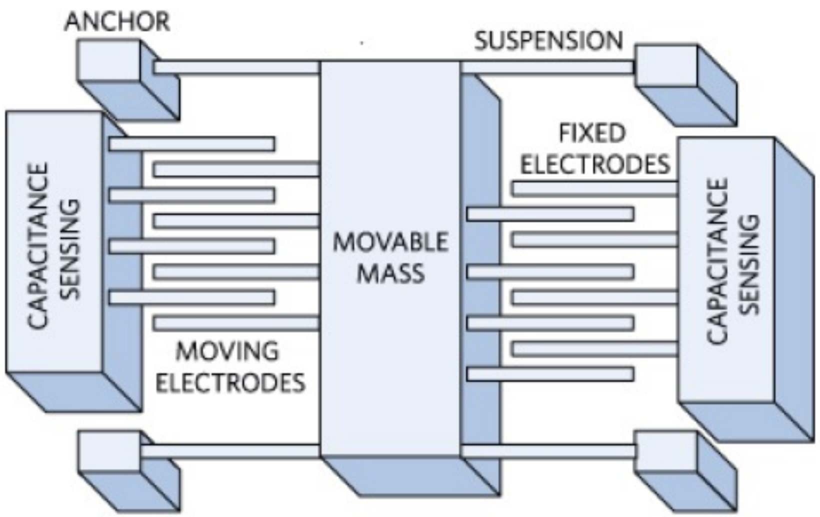


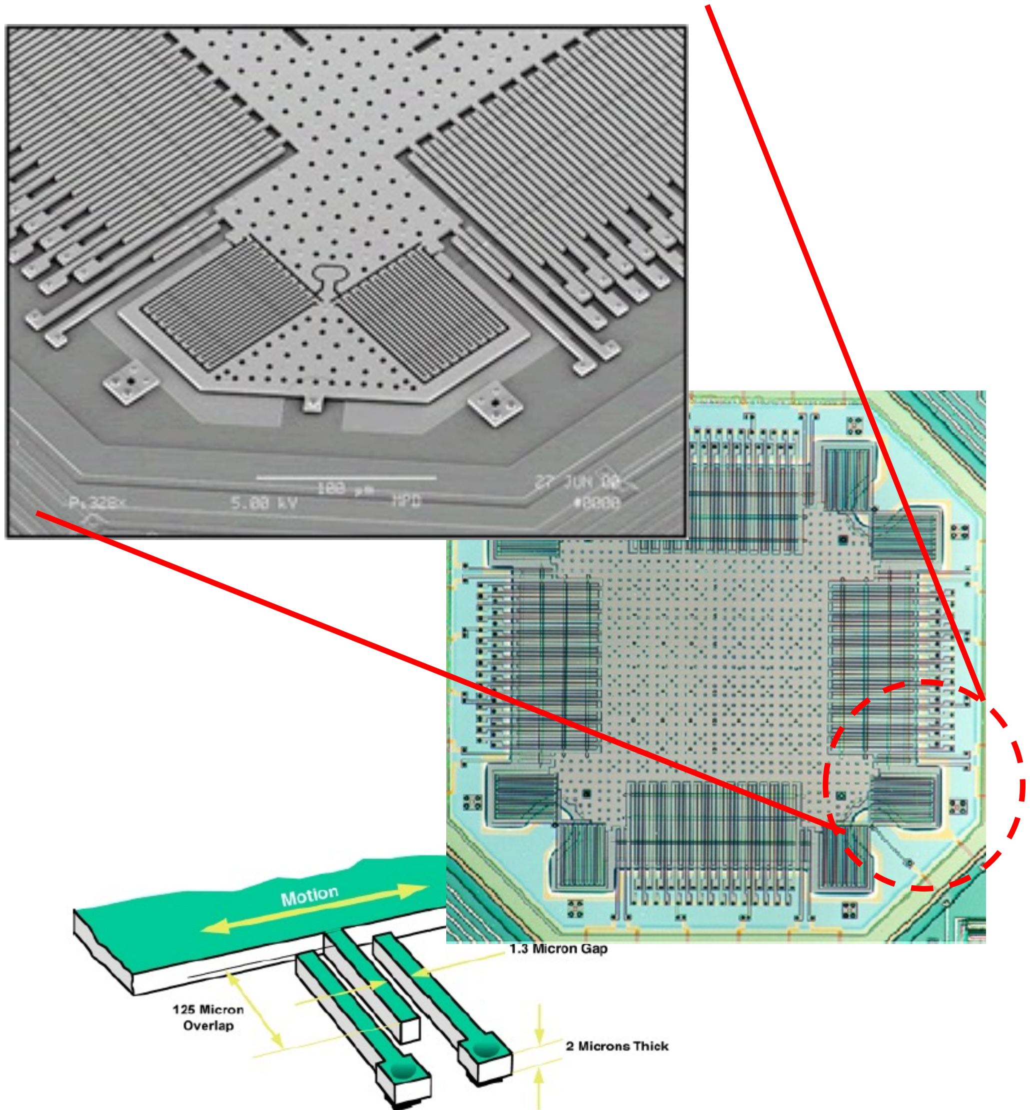
Source: Maxim Integrated

Motion Sensing - MEMS accelerometers

Capacitive MEMS accelerometer

- **The displacement of the movable mass (micrometer) is caused by acceleration.**
- **It creates an extremely small change in capacitance for proper detection. Therefore practical sensors use multiple movable and fixed electrodes, all connected in a parallel configuration as shown.**

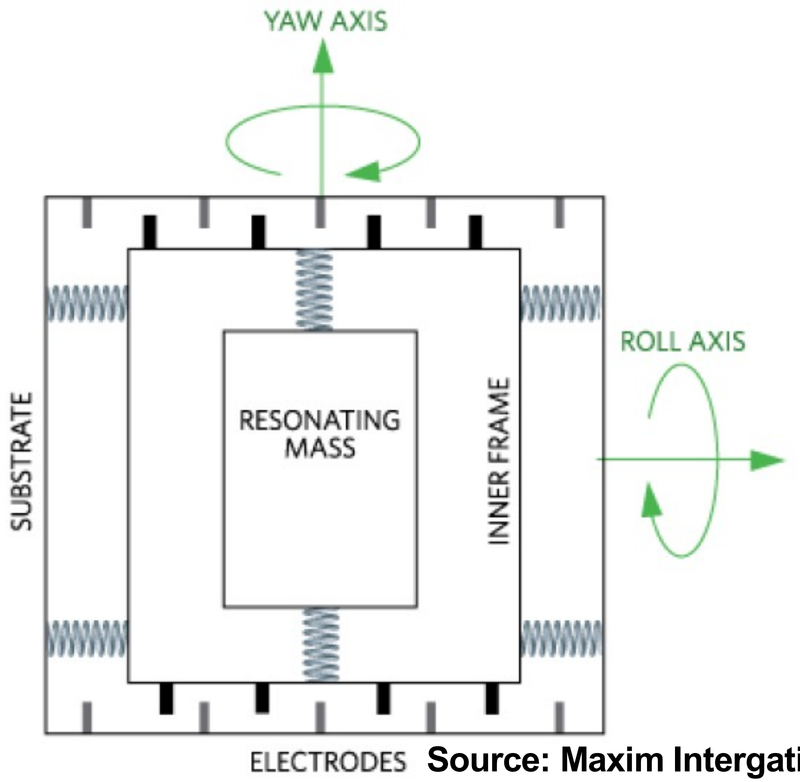




Source: Analog Devices

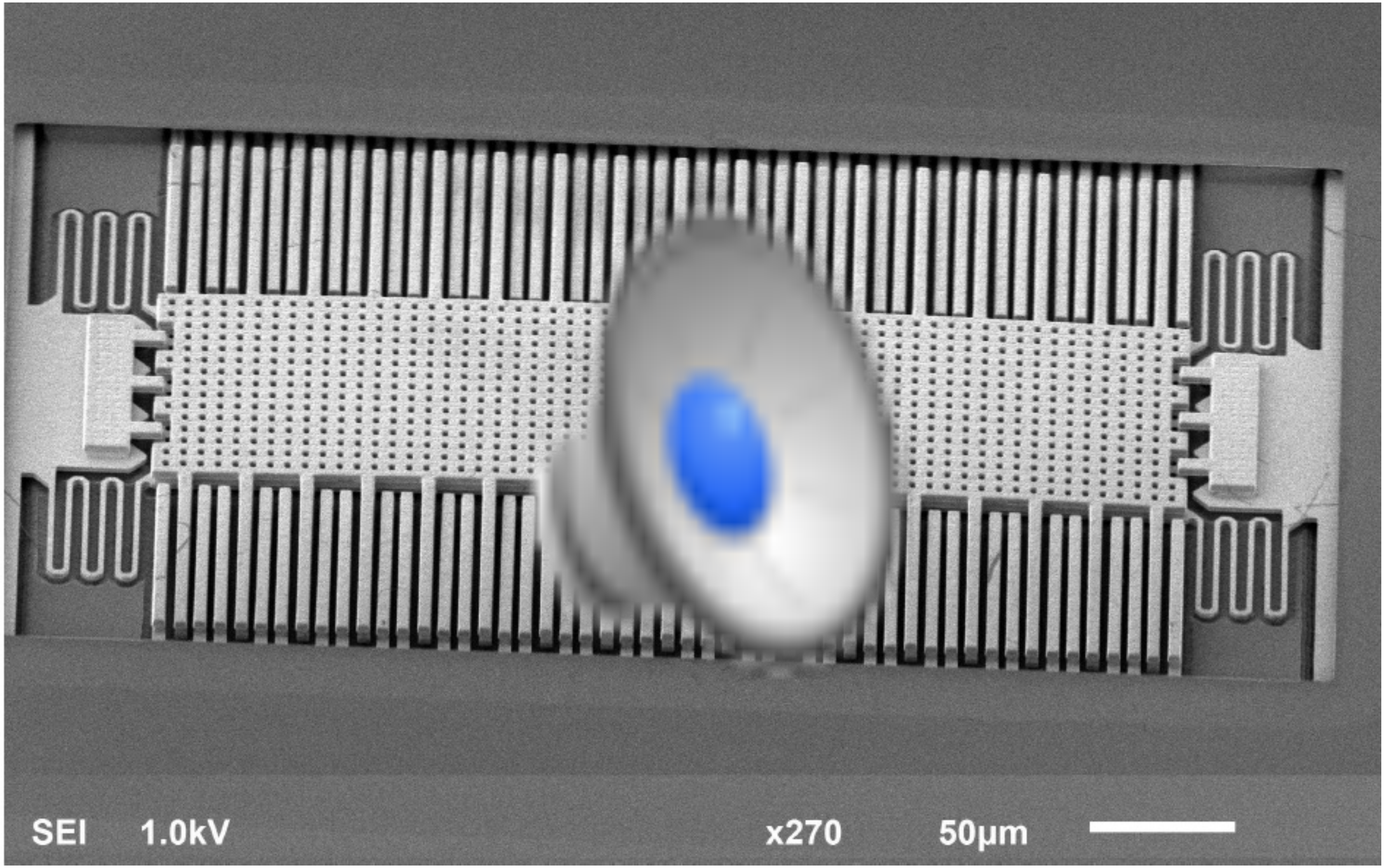
Orientation Sensing - MEMS gyroscopes

- Accelerometers measure linear acceleration (specified in mV/g) along one or several axis.
- A gyroscope measures angular velocity (specified in mV/deg/s).
- Therefore, the accelerometer's output will not respond to change in angular velocity.
- However MEMS gyroscopes are similar to accelerometer, but the structure is different as shown here.
- Here the resonating mass is mounted in an inner frame held by two springs.
- The inner frame is mounted by springs to the substrate with springs in 90 degrees to the inner springs.
- Due to the Coriolis Effect, angular rotations in the roll axis and the yaw axis (see diagram) are now translated to linear accelerations.
- The capacitive fingers are now mounted on the peripherals of the inner frame and the fixed substrate.



Source: Maxim Intergation

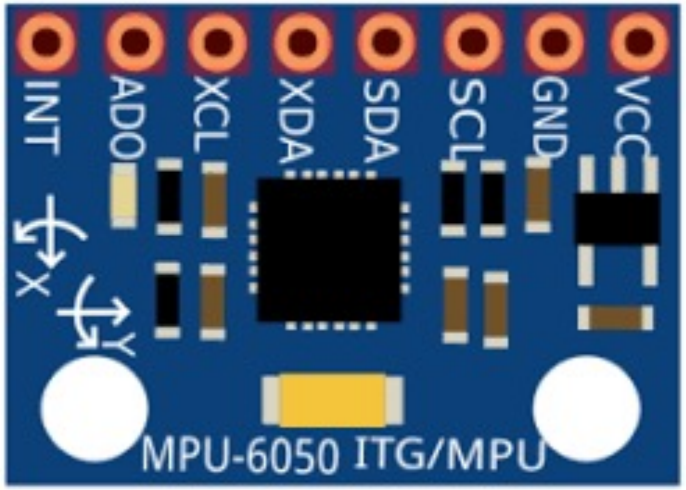
A short video on “MEMS Accelerometer”



A scanning electron microscope photo of a lateral accelerometer
Piotr Michalik et al, IEEE Senors, Nov 2015

Lab 3 – Task 1: Measuring Angel of tilt – the IMU

- The IMU – insertia measurement unit – has built in 3-axis accelerometer and 3-axis gyroscope
- The module uses I2C interface on two pins: SCL and SDA
- Easy to access from Matlab using PyBench:

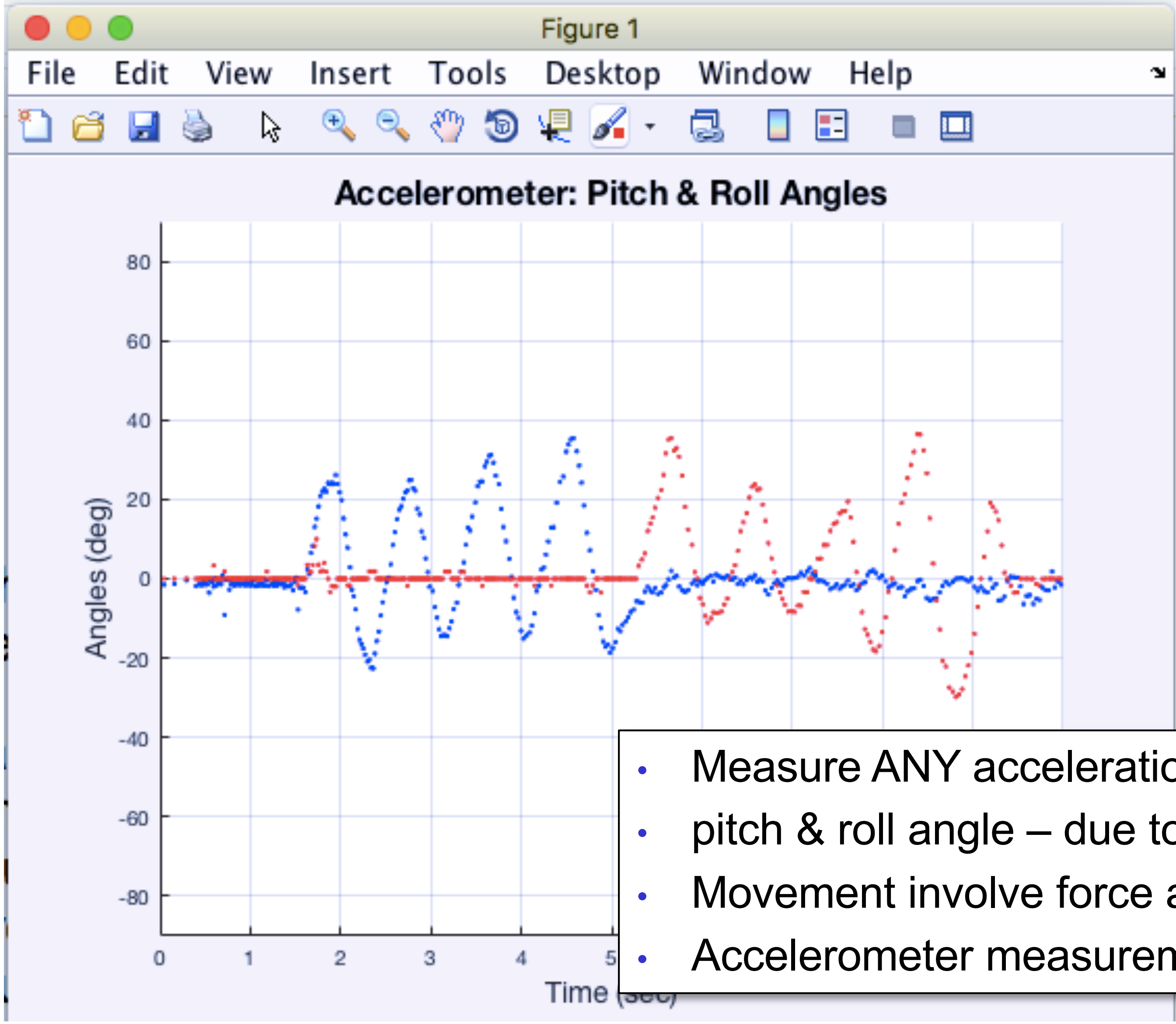



```
[p, r] = pb.get_accel();        % p, r  = pitch & roll angle in radians
[x, y, z] = pb.get_gyro();      % x, y, z = rate of rotation in 3-axes in rad/sec
```

- **Pitch angle p** – plane pointing up or down
- **Roll angle r** – plane pointing left or right
- Angle can be in unit radian or degree: degrees = radians *180 / π
- Generally use radian for calculations; use degree of display

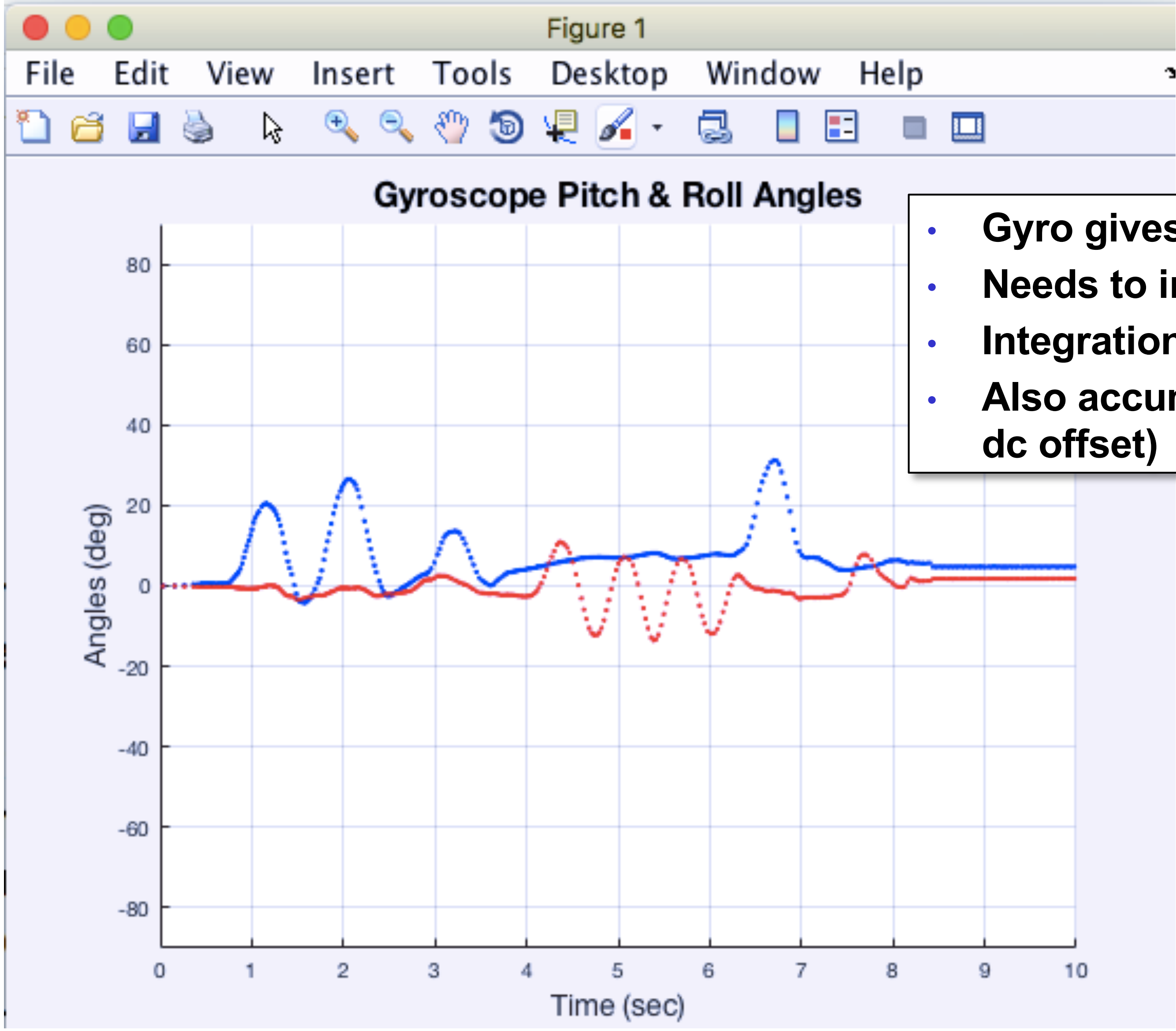
- **x, y** an **z** are the angular velocity in the three axes of rotations

Lab 3 – Task 1a: Accelerometer



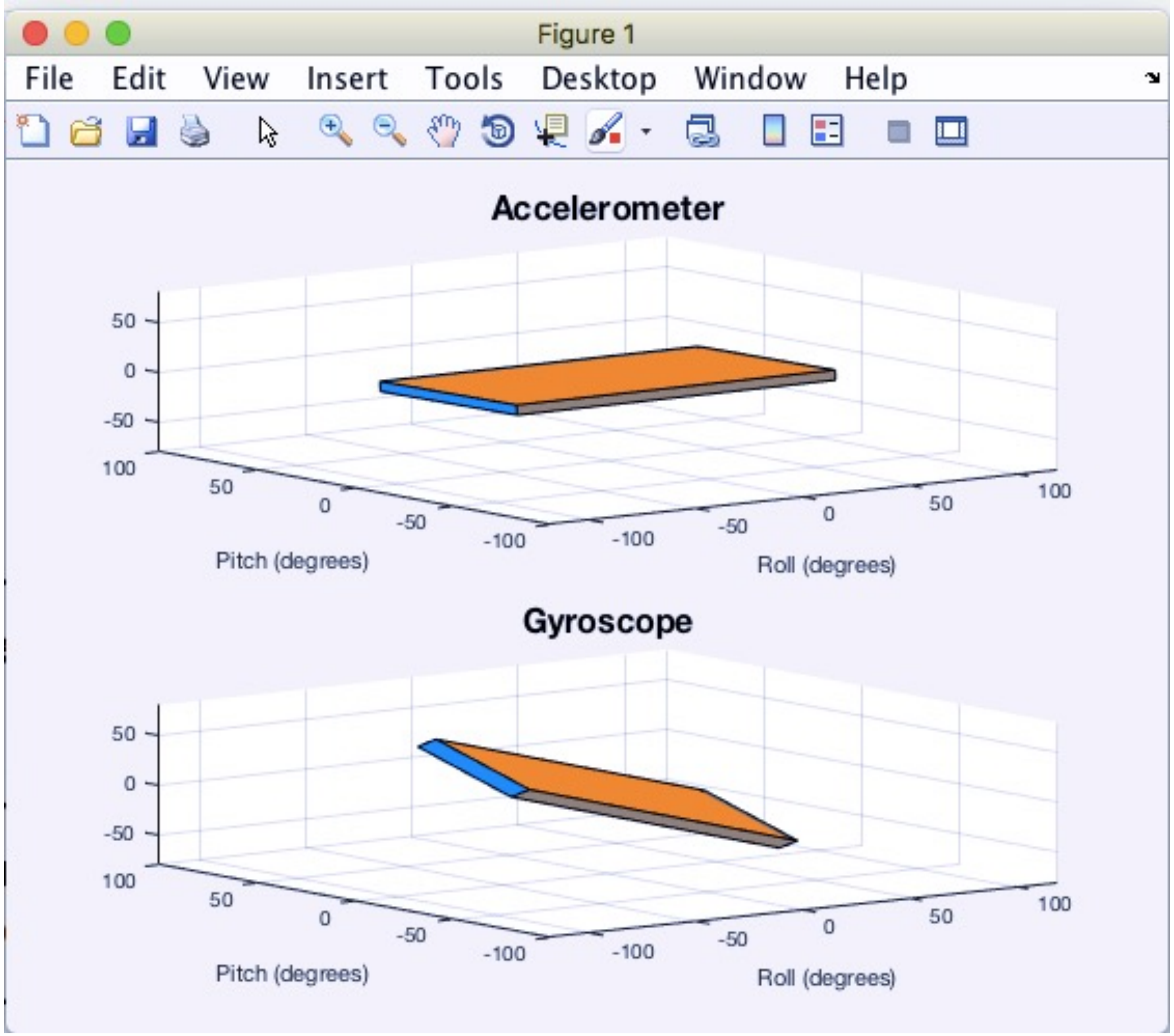
- Measure ANY acceleration
- pitch & roll angle – due to gravity g
- Movement involve force acceleration, also measured
- Accelerometer measurement of tilt angle is NOISY

Lab 3 – Task 1b: Gyroscope

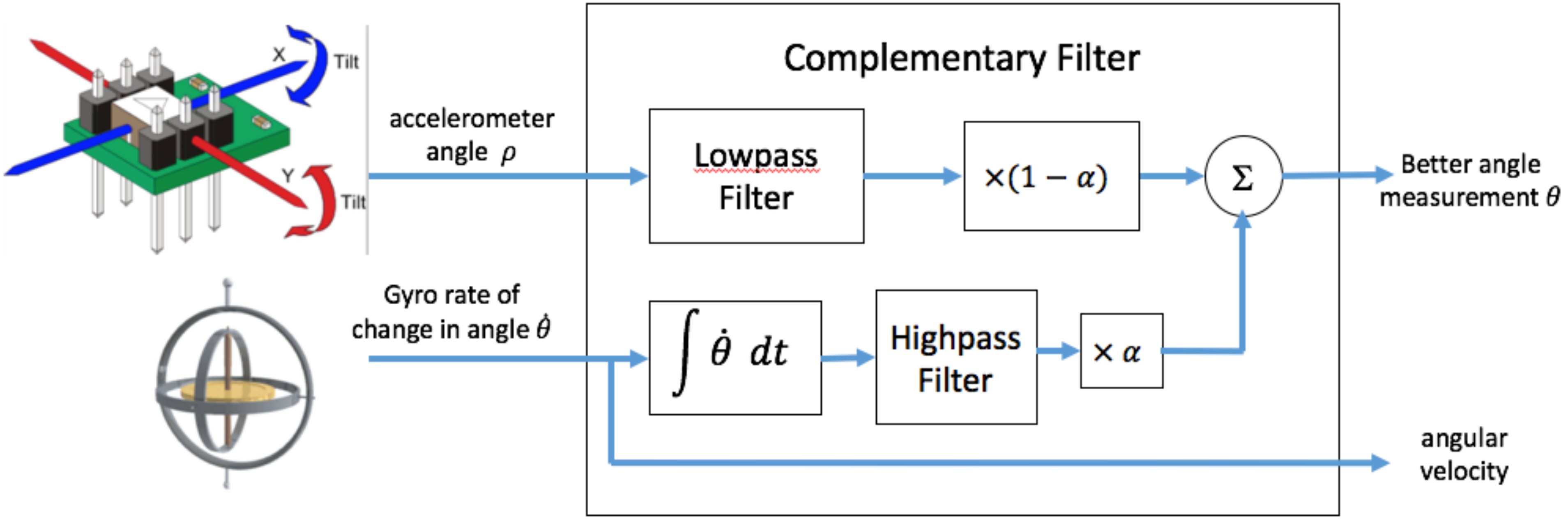


- **Gyro gives angular velocity, not angle**
- **Needs to integrate to get angle**
- **Integration = accumulation**
- **Also accumulate errors – causing drift (or dc offset)**

Lab 3 – Task 2: 3D visualization



Lab 3 – Task 3: Complementary Filter - Concept



$$\text{angle } \theta_{new} = \alpha \times \left(\theta_{old} + \dot{\theta}\, dt \right) + (1-\alpha) \times \rho$$

where

α = scaling factor chosen by users and is typically between 0.7 and 0.98

ρ = accelerometer angle

θ_{new} = new output angle

θ_{old} = previous output angle

$\dot{\theta}$ = gyroscope reading of the rate of change in angle

dt = time interval between gyro readings

Three Big Ideas

- **Accelerometer** measurement of angle is inherently **noisy** - it cannot distinguish acceleration due to gravity or due to motion.

- **Gyroscope** measurement of angle is inherently ”**drifty**” – gyroscope provides angular velocity measurement. Angle measurement is derived through integration. This results in time varying offset called drift.

- Much better angle estimation can be obtained by **filtering and fusion** of the two types of measurements.